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ON A CONJECTURE OF ERDÖS (II)

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On a conjecture of Erdös (II)

by

J. van de Lune & H.J.J. te Riele

ABSTRACT

For any integer $n \geq 2$ let $m = m(n)$ be determined by

$$(1 - \frac{1}{m})^n > \frac{1}{2} > (1 - \frac{1}{m-1})^n.$$

In this note it will be shown that

$$1^n + 2^n + \ldots + m^n > (m+1)^n$$

and

$$1^n + 2^n + \ldots + (m-1)^n < m^n$$

for almost all n. Compare the conjecture of ERDÖS stated in the *Amer. Math. Monthly*, Vol. 56 (1949), p.343 (Advanced Problem 4347).

KEY WORDS & PHRASES: *Inequalities, sums of powers of integers, uniform distribution.*

0. INTRODUCTION

In [1] ERDÖS proposed the following problem: Prove that if m and n are positive integers such that

$$(0.0)\qquad (1-\frac{1}{m})^n > \frac{1}{2} > (1-\frac{1}{m-1})^n$$

then

$$(0.1)\qquad 1^n + 2^n + \dots + (m-2)^n < (m-1)^n$$

and

$$(0.2)\qquad 1^n + 2^n + \dots + m^n > (m+1)^n.$$

Show also that

$$(0.3)\qquad 1^n + 2^n + \dots + (m-1)^n < m^n$$

in infinitely many instances and that

$$(0.4)\qquad 1^n + 2^n + \dots + (m-1)^n > m^n$$

in infinitely many instances.

The first partial solution of this problem was recently given by the first named author [4]. He showed by elementary means that (0.1) is true indeed and, in a similar fashion, he also proved the related inequality

$$(0.5)\qquad 1^n + 2^n + \dots + (m+1)^n > (m+2)^n.$$

In the meanwhile TIJDEMAN has simplified the proof of (0.1) considerably (see [4; addendum]).

In this paper we will investigate the remaining inequalities (0.2), (0.3) and (0.4).

It will be shown that the natural density of all n for which (0.2), resp. (0.3), is true is equal to 1, so that (0.3) certainly holds true in infinitely many instances. However, we have not succeeded in finding any n for which either (0.2) or (0.3) is false. Also, we have no example in which (0.4) is true.

1. PRELIMINARIES AND THE MAIN THEOREM

In [4] it was already shown that we may assume $n \geqq 2$ and that from (0.0) it follows that for any given n the number $m = m(n)$ is uniquely determined by

$$(1.1) \qquad \lambda(n) < m(n) < \lambda(n) + 1$$

or, equivalently, by

$$(1.2) \qquad m(n) = [\lambda(n)] + 1,$$

where

$$(1.3) \qquad \lambda(n) = \frac{1}{1 - 2^{-1/n}} = 1 + \frac{1}{2^{1/n} - 1} .$$

From (1.1), (1.3) and [4; lemma 3.3] it follows that

$$(1.4) \qquad m(n) > \lambda(n) = 1 + \frac{1}{\exp(\frac{\log 2}{n}) - 1} > 1 + \{\frac{n}{\log 2} - \frac{1}{2}\} > \frac{n}{\log 2}$$

so that

$$(1.5) \qquad \frac{n}{m(n)} < \log 2.$$

Also, by (1.1), (1.3) and [4; lemma 3.3] we have

$$(1.6) \qquad m(n) < 1 + \lambda(n) = 2 + \frac{1}{\exp(\frac{\log 2}{n}) - 1} < 2 + \{\frac{n}{\log 2} - \frac{1}{2} + \frac{\log 2}{12\ n}\} \leqq$$

$$\leqq \frac{n}{\log 2} + \frac{3}{2} + \frac{\log 2}{24} .$$

Since $m(2) = 4$ and

$$\frac{n}{\log 2} + \frac{3}{2} + \frac{\log 2}{24} < 2n, \qquad (n \geqq 3) \tag{1.7}$$

it follows that

$$\frac{n}{m(n)} \geqq \frac{1}{2}, \qquad (n \geqq 2). \tag{1.8}$$

Moreover, from (1.4) and (1.6) it is clear that

$$\lim_{n\to\infty} \frac{n}{m(n)} = \log 2. \tag{1.9}$$

Similarly as in [4] we define

$$\sigma_m(n) = \sum_{k=1}^{m} k^n, \qquad (m,n \in \mathbb{N}). \tag{1.10}$$

In [5] it was shown that for all $m,n \in \mathbb{N}$

$$\frac{m^{n+1}(m+1)^n}{(m+1)^{n+1} - m^{n+1}} < \sigma_m(n) < \frac{m^n(m+1)^{n+1}}{(m+1)^{n+1} - m^{n+1}} . \tag{1.11}$$

We now define $\theta = \theta(m,n)$ by

$$\sigma_m(n) = \frac{m^n(m+1)^n(m+\theta)}{(m+1)^{n+1} - m^{n+1}} \tag{1.12}$$

or, more explicitly, by

$$\theta(m,n) = -m + (m+1)\frac{\sigma_m(n)}{m^n}\{1 - (\frac{m}{m+1})^{n+1}\} \tag{1.13}$$

so that by (1.11) we have

$$0 < \theta(m,n) < 1. \tag{1.14}$$

Since (for a proof, see [5])

$$(1.15)\qquad \sigma_m(n) \geqq \frac{1}{2}\,\frac{m^{n+1}(m+1)^n + m^n(m+1)^{n+1}}{(m+1)^{n+1} - m^{n+1}} = \frac{m^n(m+1)^n(m+\frac{1}{2})}{(m+1)^{n+1} - m^{n+1}}, \qquad (m,n\in\mathbb{N})$$

we even have

$$(1.16)\qquad \frac{1}{2} \leqq \theta(m,n) < 1.$$

Concerning the function $\theta(m,n)$ we have the following

<u>(MAIN) THEOREM 1</u>. *If for* $n \geqq 2$ *the number* $m = m(n)$ *is determined by* (0.0) *then*

$$(1.17)\qquad \lim_{n\to\infty} \theta(m,n) = 2(1-\log 2).$$

Before proving this theorem we first examine the sums $\sigma_m(n)$ somewhat closer. By means of the Euler-Maclaurin summation formula we readily obtain (see [2; p.527])

$$(1.18)\qquad \sigma_m(n) = \frac{m^{n+1}}{n+1} + \frac{1}{2}m^n + \sum_{r=1}^{[\frac{n}{2}]} \frac{B_{2r}}{2r}\binom{n}{2r-1} m^{n-2r+1}$$

or, equivalently

$$(1.19)\qquad \frac{\sigma_m(n)}{m^n} = \frac{m}{n+1} + \frac{1}{2} + \sum_{r=1}^{[\frac{n}{2}]} \frac{B_{2r}}{2r}\binom{n}{2r-1} m^{-2r+1},$$

where the Bernoulli numbers B are defined by

$$(1.20)\qquad \frac{z}{e^z-1} = \sum_{r=0}^{\infty} \frac{B_r}{r!} z^r, \qquad (|z|<2\pi).$$

It is well known that for any real $\alpha \neq 0$ (see [2; p.528])

$$(1.21)\qquad \frac{1}{e^\alpha-1} = \frac{1}{\alpha} - \frac{1}{2} + \sum_{r=1}^{k} \frac{B_{2r}}{(2r)!}\alpha^{2r-1} + R_k(\alpha)$$

where

$$(1.22)\qquad R_k(\alpha) = \frac{\alpha^{2k+1}}{e^\alpha-1}\int_0^1 P_{2k+1}(x)\, e^{\alpha x}dx$$

so that

$$(1.23) \qquad \frac{1}{e^{n/m}-1} - \frac{m}{n} + \frac{1}{2} = \sum_{r=1}^{k} \frac{B_{2r}}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} + R_k\left(\frac{n}{m}\right).$$

Taking $k = [\frac{n}{2}]$ in (1.23) it follows from (1.19) and (1.23) that

$$(1.24) \qquad \left\{\frac{1}{e^{n/m}-1} - \frac{m}{n} + \frac{1}{2}\right\} - \left\{\frac{\sigma_m(n)}{m^n} - \frac{m}{n+1} - \frac{1}{2}\right\} = \frac{1}{e^{n/m}-1} + 1 - \frac{m}{n(n+1)} - \frac{\sigma_m(n)}{m^n} =$$

$$= \sum_{r=2}^{k} \frac{B_{2r}}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} \left\{1 - \frac{n(n-1)\dots(n-2r+2)}{n^{2r-1}}\right\} + R_k\left(\frac{n}{m}\right) =$$

$$= \sum_{r=2}^{k} \frac{B_{2r}}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} \delta_n(2r-2) + R_k\left(\frac{n}{m}\right)$$

where $\delta_n(\cdot)$ is defined by

$$(1.25) \qquad \delta_n(a) = 1 - \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{a}{m}\right), \qquad (a \in \mathbb{N}).$$

From (1.25) it is easily seen that for any fixed $a \in \mathbb{N}$

$$(1.26) \qquad \lim_{n\to\infty} n\, \delta_n(a) = 1 + 2 + \dots + a = \frac{1}{2}\, a(a+1).$$

Also, by mathematical induction, it is easily shown that

$$(1.27) \qquad (0 <)\ \delta_n(a) < \frac{2^a}{n}, \qquad (1 \leqq a < n;\, n \geqq 2).$$

As a consequence we have

$$(1.28) \qquad \left|\frac{B_{2r}}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} \delta_n(2r-2)\right| \leqq \frac{|B_{2r}|}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} \frac{2^{2r-2}}{n} =$$

$$= \frac{1}{2n} \frac{|B_{2r}|}{(2r)!} \left(\frac{2n}{m}\right)^{2r-1},$$

so that, in view of (1.5),

$$(1.29)\qquad \left|\frac{B_{2r}}{(2r)!}\left(\frac{n}{m(n)}\right)^{2r-1} n\,\delta_n(2r-2)\right| < \frac{1}{2}\frac{|B_{2r}|}{(2r)!}(2\log 2)^{2r-1},$$

the right hand side of (1.29) being the general term of a convergent series with positive terms (see (1.20) and note that $\log 2 < \pi$). Hence, by a uniform convergence argument (or by Lebesgue's dominated convergence theorem) we obtain

$$(1.30)\qquad \lim_{n\to\infty}\sum_{r=2}^{k}\frac{B_{2r}}{(2r)!}\left(\frac{n}{m}\right)^{2r-1} n\,\delta_n(2r-2) = \sum_{r=2}^{\infty}\frac{B_{2r}}{(2r)!}(\log 2)^{2r-1}\,\frac{1}{2}(2r-2)(2r-1) =$$

$$= \frac{1}{2}(\log 2)^2\sum_{r=2}^{\infty}\frac{B_{2r}}{(2r)!}(2r-1)(2r-2)(\log 2)^{2r-3} =$$

$$= \frac{1}{2}(\log 2)^2\,\frac{d^2}{dx^2}\left\{\sum_{r=2}^{\infty}\frac{B_{2r}}{(2r)!}x^{2r-1}\right\}_{x=\log 2}.$$

Now observe that (see [2; p.204])

$$(1.31)\qquad x\cot x = 1 - \frac{B_2}{2!}(2x)^2 + \frac{B_4}{4!}(2x)^4 - + \ldots$$

from which it is easily seen that

$$(1.32)\qquad \sum_{r=2}^{\infty}\frac{B_{2r}}{(2r)!}x^{2r-1} = \frac{i}{2}\cot\frac{ix}{2} - \frac{1}{x} - \frac{x}{12}$$

so that

$$(1.33)\qquad \frac{d^2}{dx^2}\left\{\sum_{r=2}^{\infty}\frac{B_{2r}}{(2r)!}x^{2r-1}\right\} = \frac{d^2}{dx^2}\left\{\frac{i}{2}\cot\frac{ix}{2} - \frac{1}{x} - \frac{x}{12}\right\} =$$

$$= -\frac{2}{x^3} + \frac{d}{dx}\left\{\frac{1}{4(\sin\frac{ix}{2})^2}\right\} = -\frac{2}{x^3} - \frac{e^{-x}-e^{x}}{(e^{-x/2}-e^{x/2})^4}$$

which, for $x = \log 2$, takes the value

$$(1.34)\qquad \frac{d^2}{dx^2}\left\{\frac{i}{2}\cot\frac{ix}{2} - \frac{1}{x} - \frac{x}{12}\right\}_{x=\log 2} = \frac{-2}{(\log 2)^3} + 6.$$

Hence, defining

$$(1.35) \qquad \rho(n) = n \sum_{r=2}^{k} \frac{B_{2r}}{(2r)!} \left(\frac{n}{m(n)}\right)^{2r-1} \delta_n(2r-2)$$

it follows from (1.24) that for $m = m(n)$

$$(1.36) \qquad \frac{\sigma_m(n)}{m^n} = \frac{1}{e^{n/m}-1} + 1 - \frac{m}{n(n+1)} - \frac{\rho(n)}{n} - R_k(\tfrac{n}{m})$$

where, in view of (1.30), (1.33) and (1.34)

$$(1.37) \qquad \lim_{n\to\infty} \rho(n) = \frac{1}{2}(\log 2)^2 \left\{\frac{-2}{(\log 2)^3} + 6\right\} = -\frac{1}{\log 2} + 3(\log 2)^2.$$

As to $R_k(\frac{n}{m})$ we have the following estimate

$$(1.38) \qquad |R_k(\tfrac{n}{m})| \leqq \frac{(\frac{n}{m})^{2k+1}}{e^{n/m}-1} \int_0^1 |P_{2k+1}(x)| e^{\frac{nx}{m}}\, dx.$$

Since

$$(1.39) \qquad \max_{0\leqq x\leqq 1} |P_{2k+1}(x)| \leqq \frac{4}{(2\pi)^{2k+1}}, \qquad \text{(see [2; p.527])}$$

and

$$(1.40) \qquad 2k + 1 = 2[\tfrac{n}{2}] + 1 \geqq n$$

it follows from (1.5), (1.8) and (1.38) that

$$(1.41) \qquad |R_k(\tfrac{n}{m})| \leqq \left(\frac{\log 2}{2\pi}\right)^n \frac{8}{\sqrt{e}-1}$$

so that $R_k(\frac{n}{m})$ tends exponentially fast to zero as $n \to \infty$.

As a simple consequence of (1.36), (1.37) and (1.41) we have

$$(1.42) \qquad \lim_{n\to\infty} \frac{\sigma_m(n)}{m^n} = \frac{1}{e^{\log 2}-1} + 1 = 2$$

(a relation which may also be proved by much simpler means).

PROOF OF THEOREM 1. From (1.13) it follows that

$$\theta(m,n) = m\left\{\frac{\sigma_m(n)}{m^n}\left(1 - \left(1-\frac{1}{m+1}\right)^{n+1}\right) - 1\right\} + \frac{\sigma_m(n)}{m^n}\left(1 - \left(1-\frac{1}{m+1}\right)^{n+1}\right). \tag{1.43}$$

Since

$$\lim_{n\to\infty}\left(1+\frac{\alpha(n)}{n}\right)^n = e^{\alpha} \quad \text{if} \quad \lim_{n\to\infty}\alpha(n) = \alpha \tag{1.44}$$

it follows from (1.9) and (1.42) that

$$\lim_{n\to\infty}\frac{\sigma_m(n)}{m^n}\left(1 - \left(1-\frac{1}{m+1}\right)^{n+1}\right) = 2\cdot\left(1-\frac{1}{2}\right) = 1 \tag{1.45}$$

so that, in order to determine $\lim_{n\to\infty}\theta(m,n)$, we only need to study the asymptotic behaviour of

$$\begin{aligned} m\left\{\frac{\sigma_m(n)}{m^n}\left(1 - \left(1-\frac{1}{m+1}\right)^{n+1}\right) - 1\right\} &= \\ = m\left\{\left(\frac{1}{e^{n/m}-1} + 1 - \frac{m}{n(n+1)} - \frac{\rho(n)}{n} - R_k\left(\frac{n}{m}\right)\right)\left(1 - \left(1-\frac{1}{m+1}\right)^{n+1}\right) - 1\right\} &= \\ = -m\left\{\frac{m}{n(n+1)} + \frac{\rho(n)}{n}\right\}\left\{1 - \left(1-\frac{1}{m+1}\right)^{n+1}\right\} &+ \\ + m\left\{\left(\frac{1}{e^{n/m}-1} + 1\right)\left(1 - \left(1-\frac{1}{m+1}\right)^{n+1}\right) - 1\right\} - m\,R_k\left(\frac{n}{m}\right)\left\{1 - \left(1-\frac{1}{m+1}\right)^{n+1}\right\}&. \end{aligned} \tag{1.46}$$

Since $R_k(\frac{n}{m})$ tends exponentially fast to zero as $n \to \infty$ and $m(n) = O(n)$ it follows easily that

$$\lim_{n\to\infty} m\,R_k\left(\frac{n}{m}\right)\left\{1 - \left(1-\frac{1}{m+1}\right)^{n+1}\right\} = 0. \tag{1.47}$$

Next we observe that

$$\begin{aligned} \lim_{n\to\infty} m\left\{\frac{m}{n(n+1)} + \frac{\rho(n)}{n}\right\} &= \lim_{n\to\infty}\left\{\frac{m^2}{n(n+1)} + \frac{m}{n}\rho(n)\right\} = \\ &= \frac{1}{(\log 2)^2} + \frac{1}{\log 2}\left\{-\frac{1}{\log 2} + 3\,(\log 2)^2\right\} = 3\log 2, \end{aligned} \tag{1.48}$$

so that

$$(1.49) \qquad \lim_{n\to\infty} -m\left\{\frac{m}{n(n+1)} + \frac{\rho(n)}{n}\right\}\left\{1 - (1-\frac{1}{m+1})^{n+1}\right\} = -\frac{3}{2}\log 2.$$

Finally we have

$$(1.50) \qquad m\left\{(\frac{1}{e^{n/m}-1} + 1)(1 - (1-\frac{1}{m+1})^{n+1}) - 1\right\} = m\,\frac{1 - e^{n/m}(1-\frac{1}{m+1})^{n+1}}{e^{n/m} - 1} =$$

$$= \frac{m}{e^{n/m}-1}\left\{1 - \exp(\frac{n}{m} + (n+1)\log(1-\frac{1}{m+1}))\right\} =$$

$$= -\frac{m}{e^{n/m}-1} \cdot \frac{\exp(\frac{n}{m} + (n+1)\log(1-\frac{1}{m+1})) - 1}{(0 \neq)\ \frac{n}{m} + (n+1)\log(1-\frac{1}{m+1})} \cdot \left\{\frac{n}{m} + (n+1)\log(1-\frac{1}{m+1})\right\}$$

so that, in view of

$$(1.51) \qquad \lim_{n\to\infty}\left\{\frac{n}{m} + (n+1)\log(1-\frac{1}{m+1})\right\} = \log 2 + \log\frac{1}{2} = 0$$

it follows that

$$(1.52) \qquad \lim_{n\to\infty}(1.50) = -\lim_{n\to\infty} m\left\{\frac{n}{m} + (n+1)\log(1-\frac{1}{m+1})\right\} =$$

$$= -\lim_{n\to\infty} m\left\{\frac{n}{m} - (n+1)(\frac{1}{m+1} + \frac{1}{2(m+1)^2} + 0(\frac{1}{m^3}))\right\} =$$

$$= -\lim_{n\to\infty} m\left\{\frac{n}{m} - \frac{n+1}{m+1} - \frac{n+1}{2(m+1)^2}\right\} = -\lim_{n\to\infty}\left\{\frac{n-m}{m+1} - \frac{m(n+1)}{2(m+1)^2}\right\} =$$

$$= -(\log 2 - 1 - \frac{1}{2}\log 2) = 1 - \frac{1}{2}\log 2.$$

Combining (1.45) through (1.52) with (1.43) it follows that

$$(1.53) \qquad \lim_{n\to\infty}\theta(m,n) = 1 + 0 - \frac{3}{2}\log 2 + (1 - \frac{1}{2}\log 2) = 2(1-\log 2)$$

completing the proof of the theorem. □

2. APPLICATIONS TO ERDÖS' CONJECTURE

THEOREM 2.1. *The set of all* $n \in \mathbb{N}$ *for which inequality (0.2) is false has natural density equal to zero.*

THEOREM 2.2. *The set of all* $n \in \mathbb{N}$ *for which inequality (0.3) is false has natural density equal to zero.*

Before proving these theorems we study the numbers $m(n) - \lambda(n)$ somewhat closer.

LEMMA 2.1. *If the real sequence* $\{\alpha(n)\}_{n=1}^{\infty}$ *is uniformly distributed modulo* 1 *(u.d. mod* 1*) and if* $\{\beta(n)\}_{n=1}^{\infty}$ *is any convergent real sequence then also* $\{\alpha(n) + \beta(n)\}_{n=1}^{\infty}$ *is u.d. mod* 1.

PROOF. Exercise.

LEMMA 2.2. *The (real) sequence* $\{\alpha(n)\}_{n=1}^{\infty}$ *is u.d. mod* 1 *if and only if the sequence* $\{-\alpha(n)\}_{n=1}^{\infty}$ *is u.d. mod* 1.

PROOF. Exercise.

LEMMA 2.3. *The sequence* $\{m(n) - \lambda(n)\}_{n=2}^{\infty}$ *is uniformly distributed on the interval* (0,1).

PROOF. Since $m(n) \in \mathbb{N}$ and $\lambda(n) < m(n) < \lambda(n) + 1$ it suffices to show that $\{-\lambda(n)\}_{n=2}^{\infty}$ is u.d. mod 1. In view of lemma 2.2 it therefore suffices to show that $\{\lambda(n)\}_{n=2}^{\infty}$ is u.d. mod 1.

Observing that

$$(2.1) \qquad \lambda(n) = 1 + \frac{1}{2^{1/n} - 1} = 1 + \frac{1}{\exp(\frac{\log 2}{n}) - 1} =$$

$$= 1 + \left(\frac{n}{\log 2} - \frac{1}{2} + O\left(\frac{1}{n}\right)\right) = \frac{n}{\log 2} + \frac{1}{2} + O\left(\frac{1}{n}\right), \qquad (n \to \infty)$$

it follows from lemma 2.1 and the irrationality of log 2 that $\{\lambda(n)\}_{n=2}^{\infty}$ is u.d. mod 1 (compare [3; p.92, Satz 9]), proving the lemma. □

LEMMA 2.4. *If* $\{\alpha(n)\}_{n=1}^{\infty}$ *is uniformly distributed on the interval* (0,1) *and* $\{\alpha(n_k)\}_{k=1}^{\infty}$ *is any convergent subsequence then the natural density of* $\{n_k\}_{k=1}^{\infty}$ *is equal to zero.*

PROOF. Exercise.

PROOF OF THEOREM 2.1. If (0.2) is false for only finitely many $n \in \mathbb{N}$ then we are done. Therefore, we assume (0.2) to be false for infinitely many n.

For *these* n we have

$$(2.2) \qquad 1^n + 2^n + \dots + m^n \leq (m+1)^n$$

or, equivalently,

$$(2.3) \qquad \sigma_m(n) \leq (m+1)^n.$$

Hence, writing θ instead of $\theta(m,n)$,

$$(2.4) \qquad \frac{m^n(m+1)^n(m+\theta)}{(m+1)^{n+1} - m^{n+1}} \leq (m+1)^n$$

so that

$$(2.5) \qquad m^n(m+\theta) \leq (m+1)^{n+1} - m^{n+1}$$

or, equivalently,

$$(2.6) \qquad 2 + \frac{\theta}{m} \leq \left(1 + \frac{1}{m}\right)^{n+1}$$

which may be rewritten as

$$(2.7) \qquad m \leq \frac{1}{\left(2 + \frac{\theta}{m}\right)^{(1/n+1)} - 1} .$$

From this it follows that

$$(2.8) \qquad 0 < m(n) - \lambda(n) = -1 + m(n) - \frac{1}{2^{1/n}-1} \leqq$$

$$\leqq -1 + \frac{1}{(2+\frac{\theta}{m})^{(1/n+1)}-1} - \frac{1}{2^{1/n}-1} =$$

$$= -1 + \frac{1}{\exp(\frac{1}{n+1}\log(2+\frac{\theta}{m}))-1} - \frac{1}{\exp(\frac{1}{n}\log 2)-1} =$$

$$= -1 + \left\{\frac{n+1}{\log(2+\frac{\theta}{m})} - \frac{1}{2} + 0(\frac{1}{n})\right\} - \left\{\frac{n}{\log 2} - \frac{1}{2} + 0(\frac{1}{n})\right\} =$$

$$= -1 + \frac{1}{\log(2+\frac{\theta}{m})} + n\left\{\frac{1}{\log(2+\frac{\theta}{m})} - \frac{1}{\log 2}\right\} + 0(\frac{1}{n}) =$$

$$= -1 + \frac{1}{\log(2+\frac{\theta}{m})} - \frac{n\log(1+\frac{\theta}{2m})}{\log 2\,\log(2+\frac{\theta}{m})} + 0(\frac{1}{n}).$$

In view of theorem 1 we have

$$(2.9) \qquad \lim_{n\to\infty} n\log(1+\frac{\theta}{2m}) = \lim_{n\to\infty}\log(1+\frac{\frac{\theta n}{2m}}{n})^n = \log\exp\lim_{n\to\infty}\frac{\theta n}{2m} =$$

$$= \lim_{n\to\infty}\frac{\theta n}{2m} = (1-\log 2)\cdot\log 2$$

so that, if n runs through those positive integers for which (0.2) is false, we have

$$(2.10) \qquad 0 \leqq \lim_{n\to\infty}\sup\{m(n) - \lambda(n)\} \leqq -1 + \frac{1}{\log 2} - \frac{(1-\log 2)\log 2}{(\log 2)^2} = 0$$

from which it is clear that

$$(2.11) \qquad \lim_{n\to\infty}\{m(n) - \lambda(n)\} = 0,$$

where n is such that (0.2) is false.

From this and lemmas (2.3) and (2.4) it follows that the set of all n for which (0.2) is false, has natural density equal to zero, completing the proof of theorem 2.1. □

PROOF OF THEOREM 2.2. Suppose that (0.3) is false for infinitely many $n \in \mathbb{N}$. For *these* n we have

$$1^n + 2^n + \ldots + (m-1)^n \geqq m^n \tag{2.12}$$

or, equivalently

$$\sigma_{m-1}(n) \geqq m^n. \tag{2.13}$$

Writing θ instead of $\theta(m-1,n)$ we have in view of (1.12) that

$$(m-1)^n(m-1+\theta) \geqq m^{n+1} - (m-1)^{n+1} \tag{2.14}$$

which may be rewritten as

$$m \geqq 1 + \frac{1}{(2+\frac{\theta}{m-1})^{(1/n+1)} - 1}\,. \tag{2.15}$$

It follows that

$$1 > m(n) - \lambda(n) \geqq 1 + \frac{1}{(2+\frac{\theta}{m-1})^{(1/n+1)} - 1} - (1 + \frac{1}{2^{1/n} - 1}) = \tag{2.16}$$

$$= \frac{1}{(2+\frac{\theta}{m-1})^{(1/n+1)} - 1} - \frac{1}{2^{1/n} - 1}$$

and similarly as in the proof of theorem 2.1 it follows that

$$\lim_{n\to\infty}\{m(n) - \lambda(n)\} = 1 \tag{2.17}$$

where n is such that (0.3) is false. Again, utilizing lemmas (2.3) and (2.4) this completes the proof of theorem 2.2. □

FINAL REMARK. In a forthcoming paper the first named author will demonstrate how the technique of this paper may be applied to the diophantine equation

$$1^n + 2^n + \ldots + M^n = (M+1)^n$$

or, more generally, to

$$1^n + 2^n + \ldots + M^n = G \cdot (M+1)^n$$

where G is any given positive rational number.

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